MINIMAL SURFACES IN HYPERBOLIC GEOMETRY

Winter School Côte d'Azur

Lecture I, 5th January 2026

Why hyperbolic manifolds?

Def A Riemannian manifold (M, h) is hyperbolic if g has constant sectional curvature -1

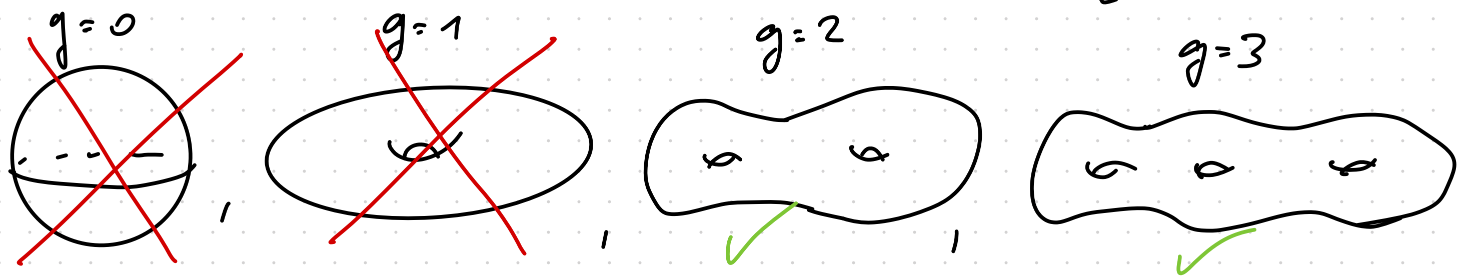
$\iff$ h is locally isometric to the hyperbolic space

unique (up to isometry) complete, simply connected, hyperbolic manifold (has many models)

Also say that "M is hyperbolic".

In dim $= 2, 3$, "most" closed manifolds are hyperbolic

- $\dim = 2$ $\quad$ M hyperbolic $\iff g \geq 2$ $\quad$ [M = connected, orientable]

$M =$ 

, ...

Gauss-Bonnet: $\chi(M) = \frac{1}{2\pi} \int_M K_g \, dA_g$

- $\dim = 3$

$\rightsquigarrow$ Thurston's Dehn surgery theorem

$\rightsquigarrow$ several random models ...

- more rare in $\dim \geq 4$!

Why minimal surfaces?

arclength parametrization

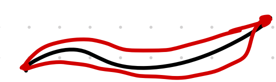
Step back: <u>geodesics</u>

Riemannian manifold

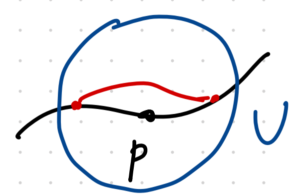
A curve $\gamma : I \text{ or } S^1 \to (M,h)$ is a *geodesic* if one of the following equivalent conditions hold:

- $\nabla^h_{\gamma'} \gamma' = 0$

- γ is a critical point of the length under smooth variations (with fixed endpoints)

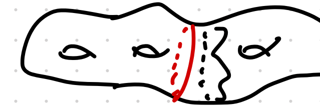


- γ is locally length-minimizing



(<u>not</u> a global minimum in general)

See Sheet 1, Exercise 4 for the equivalence



Fact: Existence and uniqueness

If (M,h) is a closed Riem. manifold with $K_h<0$, then there exists a unique closed geodesic in every nontrivial free homotopy class.

$\longleftrightarrow \; [\gamma]^{\neq 0} \in \pi_1 M / \text{conj.}$

Thm (Knieper '97, Peigné-Sambusetti '17, Besson-Courtois-Gallot, Hämenstädt ...)

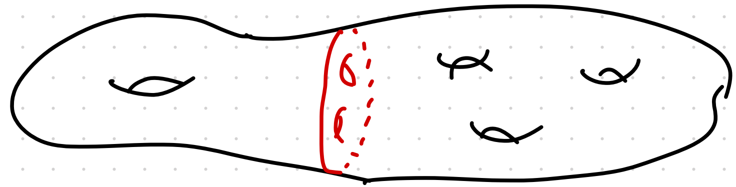
Let (M,h) be a closed Riem. manifold with $K_h \leq -1$.

Then
$$E(h) := \lim_{R\to+\infty} \frac{1}{R} \log\left(\#\left\{\begin{matrix}\text{closed geodesics}\\ \text{of length} \leq R\end{matrix}\right\}\right) \geq n-1$$

"entropy"

with $=$ $\iff$ h is hyperbolic

Idea: when $\dim M = 3$, replace closed curves by closed surfaces



Thm (Surface Subgroup Conjecture, Kahn–Markovic '12)

Let (M, h) be a closed hyperbolic 3-manifold.
Then $\pi_1 M$ contains a surface subgroup (for $g \geq 2$)

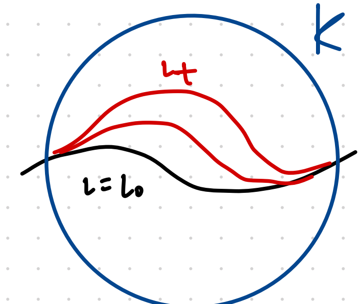
there exists $f: S_g \longrightarrow M$ continuous

such that $f_*: \pi_1 S_g \longrightarrow \pi_1 M$ is injective.

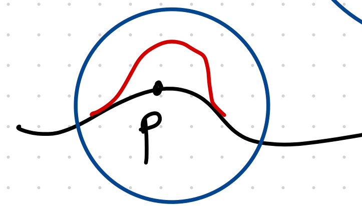
An immersion (embedding) $i: S \to (M, h)$ is <u>minimal</u> if one of the following equivalent conditions hold;

- $H \equiv 0$ mean curvature
- ι is a critical point of the area under smooth (compactly supported) variations

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Area}\,(\iota_t(S) \cap K) = 0$$



- ι is locally area-minimizing



See Sheet 1 Exercises 5 & 7 for the equivalences

Some differential geometry

$\Sigma = \iota(S)$

symmetric (0,2)-tensors

$\mathrm{I} = \iota^* h$ first fundamental form

$$\mathrm{I}(X,Y) = h(\iota_*(X), \iota_*(Y))$$

II second fundamental form

$$\nabla_{\iota_*(X)} \iota_*(Y) = \underbrace{\iota_*(\nabla^{\mathrm{I}}_X Y)}_{\in T_{\iota(p)}\Sigma} + \underbrace{\mathrm{II}(X,Y)N}_{\in (T_{\iota(p)}\Sigma)^\perp}$$

mit normal vector

(1,1)-tensor

B shape operator $\quad B(X) = -\nabla_X N$

Weingarten identity $\quad \mathrm{II}(X,Y) = \mathrm{I}(B(X),Y)$

$$H = \operatorname{tr} B = \operatorname{tr}_{I} \mathrm{II} = \mathrm{II}(e_1,e_1) + \mathrm{II}(e_2,e_2)$$

(e_1, e_2) I-orthonormal frame

e.g.

$H>0$

Rmk H changes sign if N changes sign.

$H \cdot N =$ mean curvature vector is well-defined

In fact, there exists a I-orthonormal frame such that B is diagonal:

$$\left.\begin{aligned} \mathrm{II}(e_1,e_1) &= \lambda \\ \mathrm{II}(e_2,e_2) &= \mu \end{aligned}\right\} \text{principal curvatures}$$

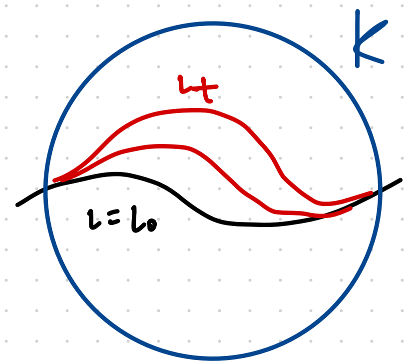
$$\mathrm{II}(e_1,e_2) = 0$$

Rmk if $B \equiv 0$, Σ is totally geodesic $\rightsquigarrow$ very rare!

First variation formula:

if $\iota_t(p) = \exp_{\iota(p)}(t f(p) N(p))$, $f \in C_0^\infty(S)$, then

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Area}(\iota_t(S)) = -\int_S f H \, dA_I$$



so, ι is critical point of area $\Leftrightarrow$ $H \equiv 0$.

("fundamental lemma of calculus of variations")

Existence results (uniqueness does not hold even for hyperbolic manifolds!)

- Schoen-Yau '79, Sacks-Uhlenbeck '81
  (+ Pitts, Federer, Meeks-Schoen-Yau...)

If M^3 is closed orientable, every π_1-injective immersion of $f: S_g \to M$, $g \geq 1$, is homotopic to a branched minimal immersion $\iota: S_g \to M$

- Osserman '70, Gulliver '73: ι is an immersion
- Freedman-Hass-Scott '83: if f is an embedding, so too is ι.
- Song '23: every closed 3-manifold contains infinitely many closed embedded minimal surfaces

Back to surface subgroups:

Thm (Kahn-Markovic '12)

Let (M, h) be a closed hyperbolic 3-manifold. There exist $c_1, c_2 > 0$ such that

$$(c_1 g)^{2g} \leq \#\left\{\begin{array}{c}\text{surface subgroups in } \pi_1 M \\ \text{of genus} \leq g \\ \text{up to conjugacy}\end{array}\right\} \leq (c_2 g)^{2g}.$$

In particular,

topological identity

$$\lim_{g \to +\infty} \frac{1}{2g \log g} \log\left(\#\left\{\begin{array}{c}\text{surface subgroups in } \pi_1 M \\ \text{of genus} \leq g \\ \text{up to conjugacy}\end{array}\right\}\right) = 1$$

Info on h can be read on the area of closed surfaces.

Thm (Calegari - Marques - Neves '22)

Let (M, h) be a closed orientable Riemannian 3-manifold with $K_h \leq -1$.

Then

$$E(h) := \lim_{\varepsilon \to 0} \liminf_{A \to +\infty} \frac{1}{A \log A} \log \left(\# \left\{ \begin{array}{c} \text{MinArea}(\Gamma): \\ \Gamma \ (1+\varepsilon) \text{ quasi-Fuchsian} \\ \text{surface subgroup in } \pi_1 M \\ \text{up to conjugacy} \end{array} \right\} \right) \geq \frac{1}{2\pi}$$

we will see later.

with equality $\iff$ h is hyperbolic

where $\text{MinArea}(\Gamma) = \min \left\{ \begin{array}{l} \text{Area}(\iota : S \to M) : \\ \iota_* (\pi_1 S) = \Gamma \end{array} \right\}$

uniquely realized if ε is small and h hyperbolic.